CHARACTERISTICS OF SELECTED MAP PROJECTIONS

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1. INTRODUCTION

This selection of map projections is based primarily on the desire to find an aesthetically pleasing equal-area projection for use in displaying statistical quantities on an all-sky map. Fair comparison of statistical quantities requires an equal-area projection. Three projections that are not equal-area (Simple Cylindrical, Mercator, Aitoff) are included for comparison purposes.

Map projections may be classified into several categories, based on the underlying geometry in which they are derived. Derivations of various projections may be found in, e.g., Pearson (1990). The categories are cylindrical, azimuthal, conical, modified forms of these, and "novelty" projections. A large selection of projections is available in Snyder and Voxland (1989), from which the projection equations (Appendix of Snyder & Voxland) and descriptive notes in the selections provided here were taken.

The section 2 reviews a few applicable aspects of differential geometry, mainly to show what is required of an equal-area map projection and to indicate the behavior of map projection meridians and parallels. Following that are the map projections, organized by type. Each map projection section contains descriptive comments, the projection equations, the coordinate derivatives (which show the behavior of the meridians and parallels), the map projection's area function, a grid plot representative of the map projection, and a map projection of a grayscale density plot.

The grid plots are a projection of a coordinate grid with 20° spacing in longitude and 10° spacing in latitude. Overlaid on the grid plots are distorted small circles. These distorted circles appear as true small circles of radius 4° on the sphere, so their apparent distortion on the mapping grid gives an indication of the distortion of the mapping.

The data for the density plot examples is from a mission simulation of the FAME spacecraft with Sun angle 45° and precession period 20 days. Specifically, the mean values of the scan angle (the angle at which the FAME telescope field of view crosses a star, with respect to the ecliptic meridian passing through the star) accumulated on a 120×96 grid of cells evenly spaced in $[\lambda, \sin\beta]$ (where λ and β are ecliptic coordinates) are shown as grayscale values. The scaling is linear, with the correspondence $[\text{black}, \text{white}] \leftrightarrow [-30^\circ, 30^\circ]$. See

http://aa.usno.navy.mil/murison/FAME/ObservationDensity/

for details. The density plots here are intended only as a means of illustrating the various map projections.

2. DIFFERENTIAL GEOMETRY OF MAP PROJECTIONS

Let the vector valued function $X : U \in \mathbb{R}^2 \rightarrow S \in \mathbb{R}^n$ be a *coordinate patch*, or *mapping*, where U is an open subset of $\mathbb{R}^2$ and S is an open subset of $M \in \mathbb{R}^2$, where M is an n-dimensional manifold. That is,

$$X(u,v) = [x_1(u,v), \ldots, x_n(u,v)] \tag{1}$$

where coordinates $(u,v) \in U$.

The *Jacobi matrix* of the patch X is

$$J(X) = \begin{bmatrix} \frac{\partial x_1}{\partial u} & \cdots & \frac{\partial x_n}{\partial u} \\ \frac{\partial x_1}{\partial v} & \cdots & \frac{\partial x_n}{\partial v} \end{bmatrix} \tag{2}$$

The Jacobi matrix can be thought of as the transformation matrix which takes tangents of curves,

$$V(u,v) = \tfrac{\partial}{\partial t}[u(t), v(t)] \tag{3}$$

into tangents to the images of curves,

$$W(x_1(u,v), \ldots, x_n(u,v)) = J(X)\,V(u,v) \tag{4}$$

under X (see, e.g., Frankel, 1997, or Gray, 1998). With regard to map projections, the elements of the Jacobi matrix of the projection mapping give us the behavior of the map projection meridians and parallels.

An example of a coordinate patch is

$$(u,v) \rightarrow [(1+\cos v)\cos u, (1+\cos v)\sin v, 2\sin v] \tag{5}$$

the *spindle torus*, shown in Figure 1.

An infinitesimal length of arc (called the *line element*) in $\mathbb{R}^2$ is $ds^2 = dx^2 + dy^2$. For a coordinate patch, the general form of the line element is

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{6}$$

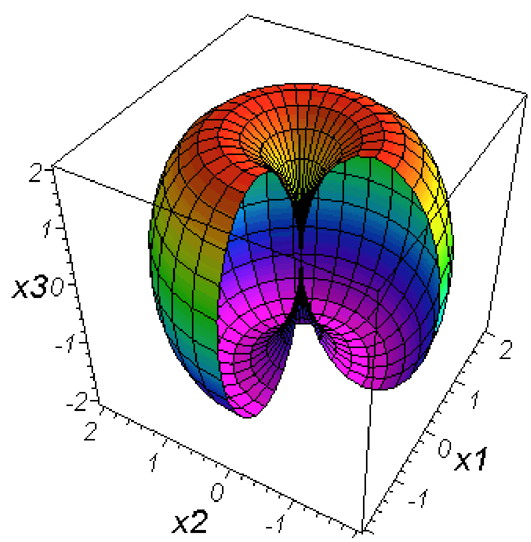

Figure 1 — An example of a coordinate patch, showing curves of constant u and constant v (cf. eq. (5)).

This is the classical notation (dating from Gauss) for a metric on a surface (see, e.g., Gray 1998). If we define $E(u,v) = \left|\frac{\partial X}{\partial u}\right|^2$, $F(u,v) = \frac{\partial X}{\partial u} \cdot \frac{\partial X}{\partial v}$, and $G(u,v) = \left|\frac{\partial X}{\partial v}\right|^2$, then the metric is a *Riemannian metric*, or the *first fundamental form*. The Riemannian metric is the inner product on the tangent space T_pM of a manifold M at a point $p = X(u,v)$. The first fundamental form gives the arc length function when integrated over a curve on the manifold. That is, if we parameterize u and v by t, then

$$s(t) = \int_0^t \sqrt{E(u,v)(\tfrac{du}{dt})^2 + 2F(u,v)\tfrac{du}{dt}\tfrac{dv}{dt} + G(u,v)(\tfrac{dv}{dt})^2}\, dt \quad (7)$$

is the arc length of a curve on M. The vectors $\frac{\partial X}{\partial u}$ and $\frac{\partial X}{\partial v}$ are tangent vectors to the surface — images under X of the coordinate tangent vectors. To find the angle between them (and hence the angles between meridians and parallels in a map projection context), we have

$$\frac{\partial X}{\partial u} \cdot \frac{\partial X}{\partial v} = \left|\frac{\partial X}{\partial u}\right|\left|\frac{\partial X}{\partial v}\right| \cos\theta \quad (8)$$

or

$$\cos\theta = \frac{F}{\sqrt{EG}} \quad (9)$$

Then

$$\sin\theta = \frac{\sqrt{EG - F^2}}{\sqrt{EG}} \quad (10)$$

The infinitesimal area element can also be found from the components of the first fundamental form. The area of the image under X an infinitesimal parallelopiped spanned by the coordinate tangent vectors is just

$$dA = \left|\frac{\partial X}{\partial u}\right|\left|\frac{\partial X}{\partial v}\right| \sin\theta\, du\, dv \quad (11)$$

Thus, from (9) and (10),

$$dA = \sqrt{EG - F^2}\, du\, dv \quad (12)$$

The area of a patch $X(U)$over a region U is then

$$A(U) = \iint_U \sqrt{EG - F^2}\, du\, dv \quad (13)$$

It can be shown (Gray, 1998) that the form of the area function, $\sqrt{EG - F^2}$, is independent of the choice of patch. Hence, it is an intrisic geometric measure.

Map projections are coordinate patches. First, we have a mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ from coordinates (λ, β) to the 3-sphere $\left\{\xi, \eta, \zeta : \sqrt{\xi^2 + \eta^2 + \zeta^2} = 1\right\}$, followed by another mapping $\Psi : \mathbb{R}^3 \to \mathbb{R}^2$ from the sphere to the map surface (x, y). The composite mapping $X = \Psi \circ \Phi$ is what we call a map projection.

If a map projection is such that an area element on the map surface (x, y) maps into an area element on the sphere with a constant scaling factor, then the map projection is said to be an *equal-area projection*. In such a projection, an area element of a given size from the map surface corresponds to the same amount of area on the sphere (multiplied by a constant scaling factor), no matter where on the map surface the area element is taken from. The area element projected from the map surface to the sphere may be both distorted in shape and rotated by $\Psi^{-1}(X)$, but the area bounded by the distorted shape is the same, independent of (x, y). The unit-radius sphere mapping is given by $(\lambda, \beta) \to [\cos\beta\cos\lambda, \cos\beta\sin\lambda, \sin\beta]$. It's area function is

$$\sqrt{EG - F^2} = |\cos\beta| \quad (14)$$

This makes intuitive sense, since for polar coordinates on a sphere of radius R an area element is $dA = R^2|\cos\beta|d\lambda d\beta$. This follows, of course, from the sphere mapping metric, $ds^2 = R^2(d\beta^2 + \cos^2\beta\, d\lambda^2)$. Thus, for a map projection to be equal-area to a sphere, its area function must be $\sqrt{EG - F^2} = a|\cos\beta|$, where a is a constant (which corresponds to the square of the equivalent sphere radius).

On the map surface, curves of constant latitude β (the map "parallels") are given by $[x, y]_\beta = \frac{\partial X}{\partial \lambda}$, while curves of constant longitude λ (the map meridians) are given by $[x, y]_\lambda = \frac{\partial X}{\partial \beta}$. Thus, we have the following properties of the parallels and meridians, based on the behavior of the mapping's coordinate derivatives (i.e., the mapping's Jacobi matrix elements):

- Parallels are straight lines: $\frac{\partial y}{\partial \lambda} = 0$
- Parallels are equally spaced: $\frac{\partial y}{\partial \beta} = const$
- Meridians are straight lines: $\frac{\partial x}{\partial \beta} = 0$
- Meridians are equally spaced: $\frac{\partial x}{\partial \lambda} = const$

3. CYLINDRICAL PROJECTIONS

3.1. Simple Cylindrical Projection

- Origin: Eratosthenes (275?-195? B.C), and, independently, Marinus of Tyre (~100 A.D.).
- aka: Plate Carree projection.
- Meridians: equally spaced straight parallel lines.
- Parallels: equally spaced straight parallel lines.
- Scale: true along equator and along all meridians, increases with distance from equator along parallels, constant along any given parallel.
- Distortion: both shape and area, increasing with distance from equator along parallels.
- Mapping function: $[x, y] = [\lambda, \beta]$
- Coordinate derivatives:

$$\frac{\partial X}{\partial \lambda} = [1, 0]$$

$$\frac{\partial X}{\partial \beta} = [0, 1]$$

- Area function: $\sqrt{EG - F^2} = 1$

- Simple Cylindrical coordinate grid:

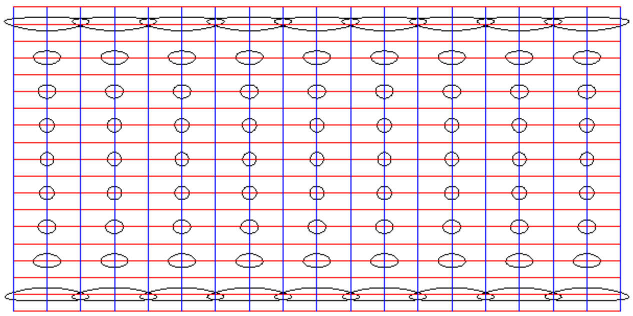

Figure 2 — Simple Cylindrical projection.

- Simple Cylindrical projected density plot:

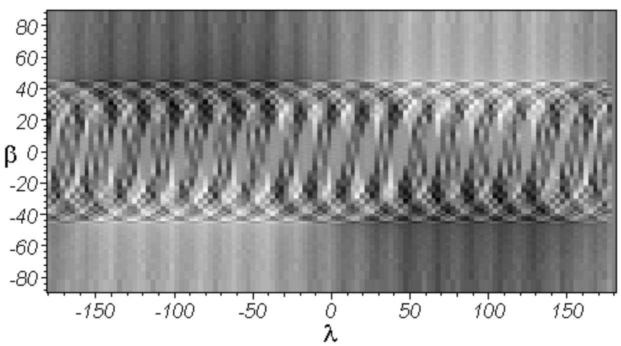

Figure 3 — Simple Cylindrical projection of density plot.

3.2. Mercator Projection

- Origin: Mercator, 1569.
- aka: Wright projection.
- Meridians: equally spaced straight parallel lines.
- Parallels: unequally spaced straight parallel lines, closest near equator.
- Scale: true along equator or along two parallels equidistant from equator, increases with distance from equator to infinity at the poles, constant along any given parallel.
- Distortion: conformal (local angle preservation), increases with distance from equator along parallels.
- Mapping function: $[x, y] = [\lambda, \ln\tan\left(\frac{\beta}{2} + \frac{\pi}{4}\right)]$
- Coordinate derivatives:

$$\frac{\partial X}{\partial \lambda} = [1, 0]$$

$$\frac{\partial X}{\partial \beta} = \left[0, \frac{1 + \tan^2\left(\frac{\beta}{2} + \frac{\pi}{4}\right)}{2\tan\left(\frac{\beta}{2} + \frac{\pi}{4}\right)}\right]$$

- Area function: $\sqrt{EG - F^2} = \dfrac{1 + \tan^2\left(\frac{\beta}{2} + \frac{\pi}{4}\right)}{2\left|\tan\left(\frac{\beta}{2} + \frac{\pi}{4}\right)\right|}$

- Mercator coordinate grid:

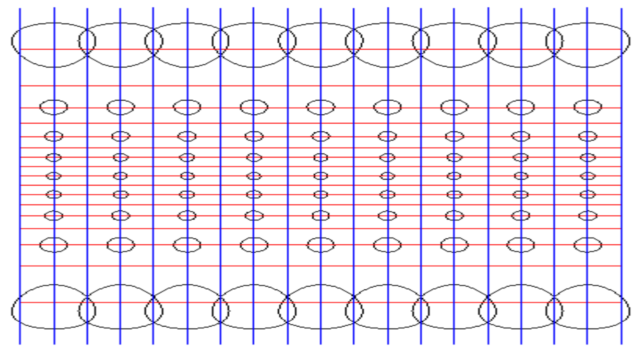

Figure 4 — Mercator projection.

- Mercator projected density plot:

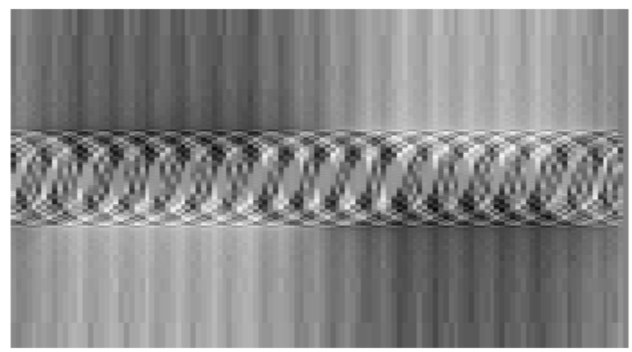

Figure 5 — Mercator projection of density plot.

3.3. Lambert Cylindrical Equal-Area Projection

- Origin: J.H. Lambert, 1772.
- aka: Cylindrical Equal-Area projection.
- Meridians: equally spaced straight parallel lines.
- Parallels: unequally spaced straight parallel lines, farthest apart near equator.
- Scale: true along equator, increases with distance from equator along parallels, decreases with distance from equator along meridians (thus maintaining equal area).
- Distortion: shape distortion in polar regions is extreme.
- Mapping function: $[x, y] = [\lambda, \sin\beta]$
- Coordinate derivatives:

$$\frac{\partial X}{\partial \lambda} = [1, 0]$$

$$\frac{\partial X}{\partial \beta} = [0, \cos\beta]$$

- Area function: $\sqrt{EG - F^2} = |\cos\beta|$

- Lambert Cylindrical coordinate grid:

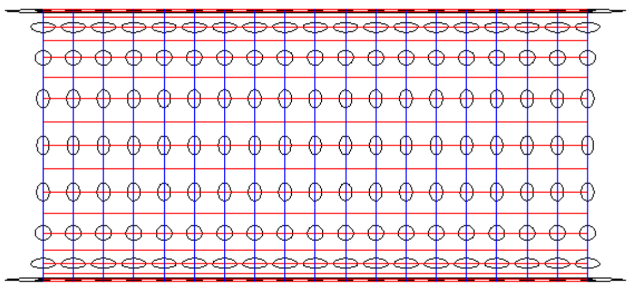

Figure 6 — Lambert Cylindrical Equal-Area projection.

- Lambert Cylindrical projected density plot:

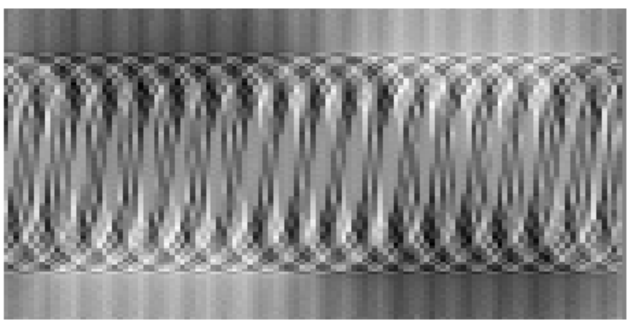

Figure 7 — Lambert Cylindrical Equal-Area projection of density plot.

4. PSUEDO-CYLINDRICAL PROJECTIONS

4.1. Sinusoidal Equal-Area Projection

- Origin: Developed in 16th century.
- aka: Sanson-Flamsteed, Mercator Equal-Area.
- Meridians: equally spaced (along any given parallel) sinusoidal curves intersecting at the poles.
- Parallels: equally spaced straight parallel lines.
- Scale: true along every parallel and along the central meridian.
- Distortion: severe near outer meridians at high latitudes, zero along equator and along central meridian.
- Mapping function: $[x,y]=[\lambda\cos\beta,\beta]$
- Coordinate derivatives:

$$\frac{\partial X}{\partial\lambda}=[\cos\beta,0]$$

$$\frac{\partial X}{\partial\beta}=[-\lambda\sin\beta,1]$$

- Area function: $\sqrt{EG-F^2}=|\cos\beta|$

- Sinusoidal coordinate grid:

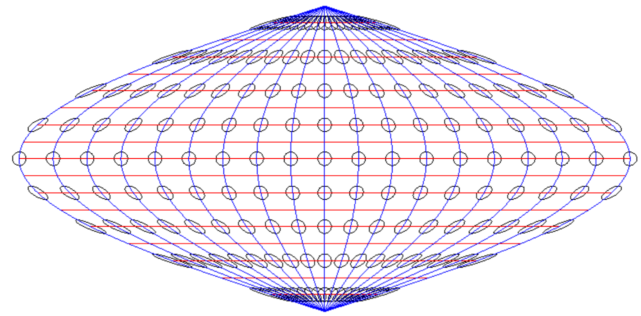

Figure 8 — Sinusoidal Equal-Area projection.

- Sinusoidal projected density plot:

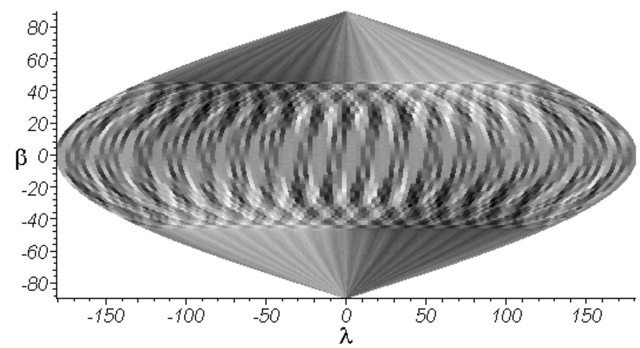

Figure 9 — Sinusoidal Equal-Area projection of density plot.

4.2. Craster Parabolic Equal-Area Projection

- Origin: J.E.E. Craster, 1929.
- aka: Putnins P4 (independently in 1934).
- Meridians: equally spaced (along any given parallel) parabolas intersecting at the poles.
- Parallels: unequally spaced straight parallel lines, farthest apart near equator, spacing changes very gradually.
- Scale: true along latitudes 36°46' N. and S., constant along any given latitude.
- Distortion: severe near outer meridians at high latitudes but somewhat less than that of the Sinusoidal, zero only at latitudes 36°46' N. and S. on the central meridian.
- Mapping function: $[x,y]=\left[\lambda\left(2\cos\frac{2\beta}{3}-1\right),\pi\sin\frac{\beta}{3}\right]$
- Coordinate derivatives:

$$\frac{\partial X}{\partial\lambda}=\left[2\cos\frac{2\beta}{3}-1,0\right]$$

$$\frac{\partial X}{\partial\beta}=\left[-\frac{4}{3}\lambda\sin\frac{2\beta}{3},\frac{\pi}{3}\cos\frac{\beta}{3}\right]$$

- Area function: $\sqrt{EG-F^2}=\frac{\pi}{3}|\cos\beta|$

Both x and y have been multiplied by a scale factor of $\sqrt{\pi/3}$ so that the $\lambda=\pi$ meridian intersects the equator at $x=\pi$, as do those of both the Quartic Authalic and the Sinusoidal projections, and so that the pole is at $y=\pi/2$, as is that of the Sinusoidal projection. One consequence of this scale factor is that the area element is not $|\cos\beta|$, as are the area elements of the Sinusoidal and Quartic Authalic projections. Removing the scale factor changes the area element back to $|\cos\beta|$. Another probable consequence is that the latitudes of true scale might be shifted slightly. The scale factor allows easier comparison (see below) with the Sinusoidal and Quartic Authalic projections.

- Craster Parabolic coordinate grid:

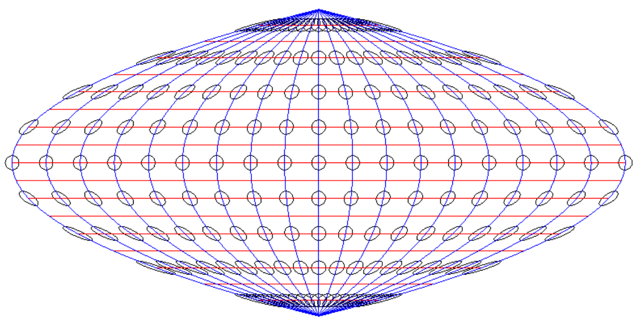

Figure 10 — Craster Parabolic Equal-Area projection.

- Craster Parabolic projected density plot:

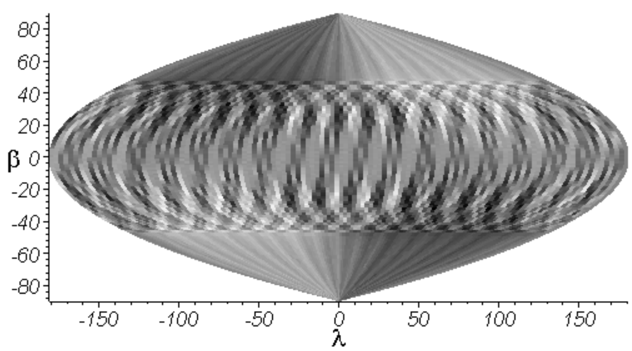

Figure 11 — Craster Parabolic Equal-Area projection of density plot.

4.3. Quartic Authalic Equal-Area Projection

- Origin: K. Siemon, 1937; independently by O.S. Adams, 1945.
- Meridians: equally spaced (along any given parallel) quartics intersecting at the poles.
- Parallels: unequally spaced straight parallel lines, farthest apart near equator, spacing changes gradually.
- Scale: true along equator, constant along any given latitude.
- Distortion: severe near outer meridians at high latitudes but somewhat less than that of the Sinusoidal, zero along equator.
- Mapping function: $[x, y] = \left[\lambda \dfrac{\cos\beta}{\cos\dfrac{\beta}{2}}, 2\sin\dfrac{\beta}{2}\right]$
- Coordinate derivatives:

$$\frac{\partial X}{\partial \lambda} = \left[\frac{\cos\beta}{\cos\dfrac{\beta}{2}}, 0\right]$$

$$\frac{\partial X}{\partial \beta} = \left[-\lambda \frac{(2+\cos\beta)\sin\beta}{4\cos^3\dfrac{\beta}{2}}, \cos\frac{\beta}{2}\right]$$

- Area function: $\sqrt{EG - F^2} = |\cos\beta|$
- Quartic Authalic coordinate grid:

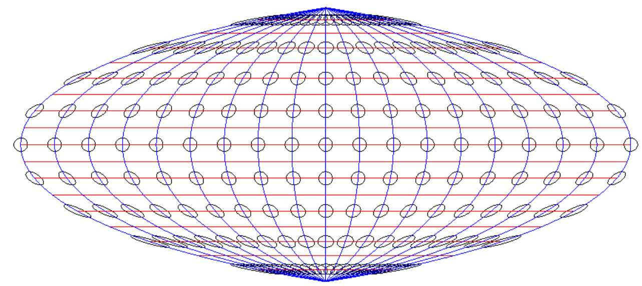

Figure 12 — Quartic Authalic Equal-Area projection.

- Quartic Authalic projected density plot:

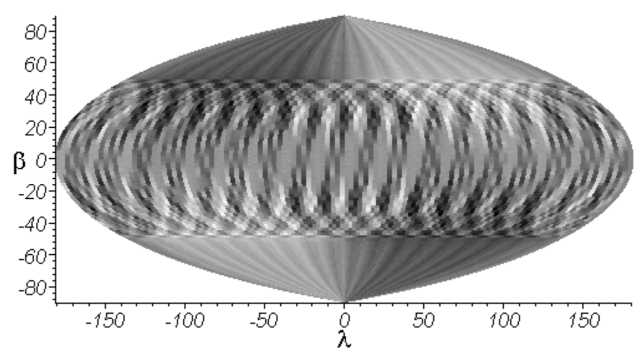

Figure 13 — Quartic Authalic Equal-Area projection of density plot.

Figure 14 is a combined plot of one quadrant of the Quartic Authalic (blue), Craster Parabolic (red), and Sinusoidal (black) projections. Theses are similar equal-area projections. The Sinusoidal is most peaked at the poles; the Quartic Authalic is most "squashed" or elliptical. For clarity, Sinusoidal distortion circles are not plotted above 40 degrees, and the distortion circles of the other cases are not plotted above 60 degrees. Distortion on the outer meridians at high latitudes is very similar in the Quartic Authalic and Craster Parabolic projections and slightly less than that of the Sinusoid projection.

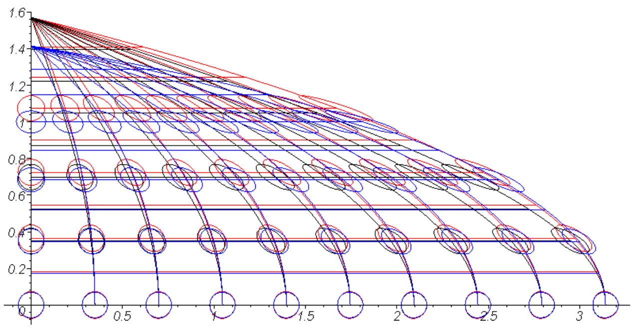

Figure 14 — Comparison of one quadrant of Pseudo-Cylindrical Equal-Area projections: Quartic Authalic (blue), Craster Parabolic (red), and Sinusoidal (black).

5. AZIMUTHAL PROJECTIONS

5.1. Lambert Azimuthal Equal-Area Projection

- Origin: J.H. Lambert, 1772.
- aka: Lorgna (independently, 1789), Zenithal Equal-Area, Zenithal Equivalent.
- Meridians: unequally spaced complex curves intersecting at the poles.
- Parallels: unequally spaced complex curves.
- Scale: true only at center, decreases with distance from center along radii, increases with distance from center perpendicular to radii.
- Distortion: moderate for one hemisphere but extreme for the whole sphere.
- Mapping function:

$$[x,y] = \left[\frac{\sqrt{2}\,\cos\beta\sin\lambda}{\sqrt{1+\cos\beta\cos\lambda}},\ \frac{\sqrt{2}\,\sin\beta}{\sqrt{1+\cos\beta\cos\lambda}} \right]$$

- Coordinate derivatives:

$$\frac{\partial X}{\partial \lambda} = \begin{bmatrix} \dfrac{[(1+\cos^2\lambda)\cos\beta + 2\cos\lambda]\cos\beta}{\sqrt{2}\,(1+\cos\beta\cos\lambda)^{3/2}} \\[2ex] \dfrac{\sin\beta\cos\beta\sin\lambda}{\sqrt{2}\,(1+\cos\beta\cos\lambda)^{3/2}} \end{bmatrix}$$

$$\frac{\partial X}{\partial \beta} = \begin{bmatrix} -\dfrac{(2+\cos\beta\cos\lambda)\sin\beta\sin\lambda}{\sqrt{2}\,(1+\cos\beta\cos\lambda)^{3/2}} \\[2ex] \dfrac{(1+\cos^2\beta)\cos\lambda + 2\cos\beta}{\sqrt{2}\,(1+\cos\beta\cos\lambda)^{3/2}} \end{bmatrix}$$

- Area function: $\sqrt{EG-F^2} = |\cos\beta|$
- Lambert Azimuthal coordinate grid (one hemisphere, $\Delta\lambda = 10$ deg):

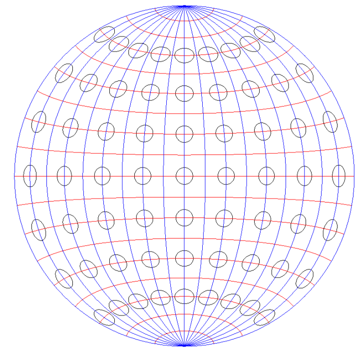

Figure 15 — Lambert Azimuthal Equal-Area projection.

- Lambert Azimuthal projected density plot (one hemisphere):

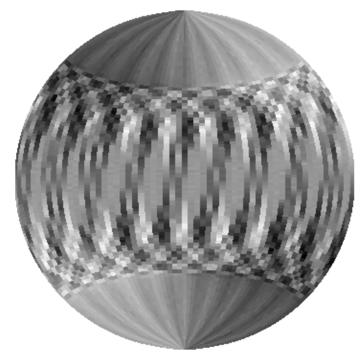

Figure 16 — Lambert Azimuthal Equal-Area projection of density plot.

- Lambert Azimuthal coordinate grid (full sphere, $\Delta\lambda = 20$ deg):

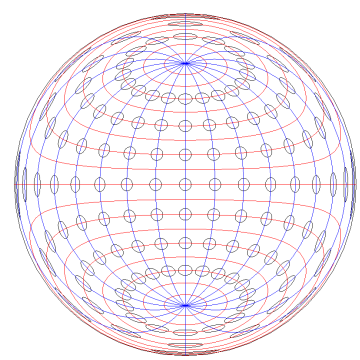

Figure 17 — Lambert Azimuthal Equal-Area projection.

6. MODIFIED AZIMUTHAL PROJECTIONS

6.1. Aitoff Projection

- Origin: D. Aitoff, 1889.
- Meridians: complex curves, equally spaced along equator, intersecting at the poles.
- Parallels: complex curves, equally spaced along central meridian, concave towards nearest pole.
- Scale: true along equator and central meridian.

- Distortion: moderate shape and area distortion.
- Mapping function:

$$[x,y] = \begin{bmatrix} \dfrac{4}{\pi}\cos^{-1}\left(\cos\beta\cos\dfrac{\lambda}{2}\right)\sqrt{1-\dfrac{\sin^2\beta}{1-\cos^2\beta\cos^2\dfrac{\lambda}{2}}} \\ \dfrac{2\cos^{-1}\left(\cos\beta\cos\dfrac{\lambda}{2}\right)\sin\beta}{\pi\sqrt{1-\cos^2\beta\cos^2\dfrac{\lambda}{2}}} \end{bmatrix}$$

- Coordinate derivatives:

$$\frac{\partial X}{\partial\lambda} = \begin{bmatrix} \dfrac{2\left|\sin\dfrac{\lambda}{2}\right||\cos\beta|\cos\beta\sin\dfrac{\lambda}{2}}{\pi\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)} + \dfrac{2\cos^{-1}\left(\cos\beta\cos\dfrac{\lambda}{2}\right)\sin^2\beta\cos^2\beta\cos\dfrac{\lambda}{2}\sin\dfrac{\lambda}{2}}{\pi\sqrt{1-\dfrac{\sin^2\beta}{1-\cos^2\beta\cos^2\dfrac{\lambda}{2}}}\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)^2} \\ \dfrac{\cos\beta\sin\beta\sin\dfrac{\lambda}{2}}{\pi\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)} - \dfrac{\sin\beta\cos^2\beta\cos\dfrac{\lambda}{2}\sin\dfrac{\lambda}{2}\cos^{-1}\left(\cos\beta\cos\dfrac{\lambda}{2}\right)}{\pi\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)^{3/2}} \end{bmatrix}$$

$$\frac{\partial X}{\partial\beta} = \begin{bmatrix} \dfrac{4\left|\sin\dfrac{\lambda}{2}\right|\sin\beta\cos\dfrac{\lambda}{2}|\cos\beta|}{\pi\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)} - \dfrac{4\sin\beta\sin^2\dfrac{\lambda}{2}\cos\beta\cos^{-1}\left(\cos\beta\cos\dfrac{\lambda}{2}\right)}{\pi|\cos\beta|\left|\sin\dfrac{\lambda}{2}\right|\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)^{3/2}} \\ \dfrac{2(\sin\beta)^2\cos\dfrac{\lambda}{2}}{\pi\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)} + \dfrac{2\cos\beta\sin^2\dfrac{\lambda}{2}\cos^{-1}\left(\cos\beta\cos\dfrac{\lambda}{2}\right)}{\pi\left(1-\cos^2\beta\cos^2\dfrac{\lambda}{2}\right)^{3/2}} \end{bmatrix}$$

- Area function:

$$\sqrt{EG-F^2} = \frac{4}{\pi^2}\frac{\left|\cos^{-1}\left(\cos\beta\cos\dfrac{\lambda}{2}\right)\right|}{\sqrt{1-\cos^2\beta\cos^2\dfrac{\lambda}{2}}}|\cos\beta|$$

- Aitoff coordinate grid:

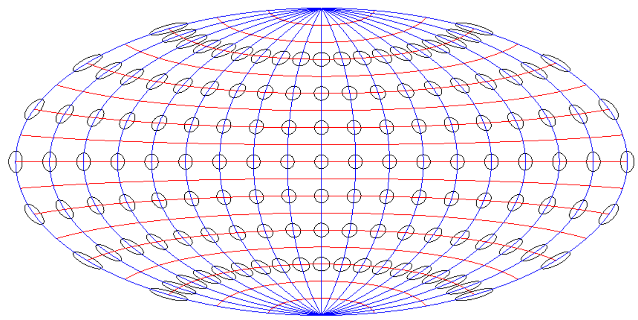

Figure 18 — Aitoff projection.

- Aitoff projected density plot:

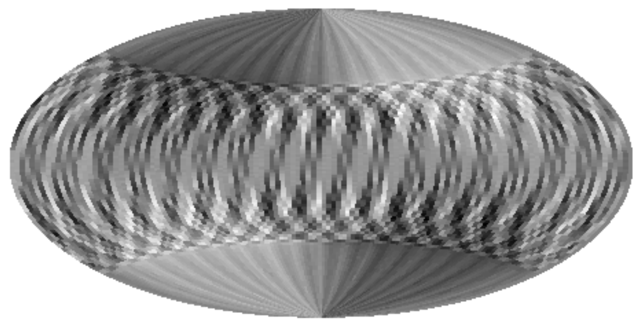

Figure 19 — Aitoff projection of density plot.

6.2. Hammer Equal-Area Projection

- Origin: H.H. Ernst von Hammer, 1892 (inspired by Aitoff's projection).
- aka: Hammer-Aitoff projection. Often erroneously called Aitoff projection.
- Meridians: complex curves, unequally spaced along equator, intersecting at the poles.
- Parallels: complex curves, unequally spaced along central meridian, concave towards nearest pole.
- Scale: decreases from center along equator, decreases from center along central meridian.
- Distortion: moderate, less shearing on outer meridians near poles than with pseudocylindrical projections.
- Mapping function:

$$[x,y] = \left[\frac{2\cos\beta\sin\dfrac{\lambda}{2}}{\sqrt{1+\cos\beta\cos\dfrac{\lambda}{2}}}, \frac{\sin\beta}{\sqrt{1+\cos\beta\cos\dfrac{\lambda}{2}}}\right]$$

- Coordinate derivatives:

$$\frac{\partial X}{\partial \lambda} = \begin{bmatrix} \dfrac{\left[(3+\cos\lambda)\cos\beta + 4\cos\dfrac{\lambda}{2}\right]\cos\beta}{4\left(1+\cos\beta\cos\dfrac{\lambda}{2}\right)^{3/2}} \\[2ex] \dfrac{\sin\beta\cos\beta\sin\dfrac{\lambda}{2}}{4\left(1+\cos\beta\cos\dfrac{\lambda}{2}\right)^{3/2}} \end{bmatrix}$$

$$\frac{\partial X}{\partial \beta} = \begin{bmatrix} -\dfrac{\left(\sin\lambda\cos\beta + 4\sin\dfrac{\lambda}{2}\right)\sin\beta}{2\left(1+\cos\beta\cos\dfrac{\lambda}{2}\right)^{3/2}} \\[2ex] \dfrac{(1+\cos^2\beta)\cos\dfrac{\lambda}{2} + \cos\beta}{2\left(1+\cos\beta\cos\dfrac{\lambda}{2}\right)^{3/2}} \end{bmatrix}$$

- Area function: $\sqrt{EG-F^2} = \frac{1}{2}|\cos\beta|$
- Hammer coordinate grid:

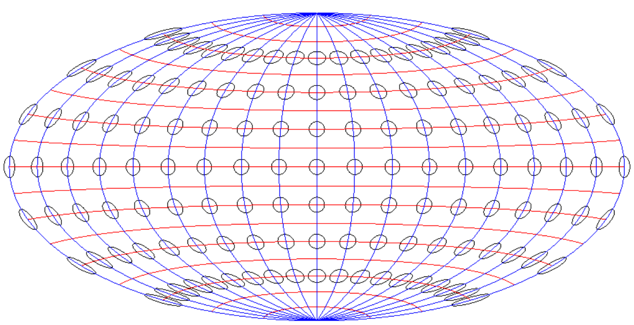

Figure 20 — Hammer Equal-Area projection.

- Hammer projected density plot:

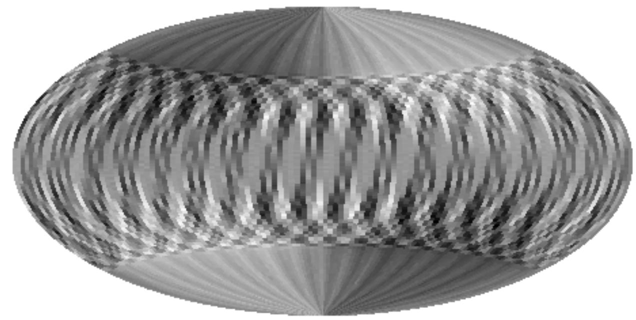

Figure 21 — Hammer Equal-Area projection of density plot.

Figure 22 is a plot of one quadrant of the Aitoff (red) and Hammer (black) projections. The unequal spacings of the Hammer meridians along the equator and the Hammer parallels along the central meridian are evident. These unequal spacings follow from the requirement of equal area. Distortion on the outer meridians at high latitudes is somewhat less severe with the Aitoff projection. It is unfortunate that the Aitoff projection is not equal-area.

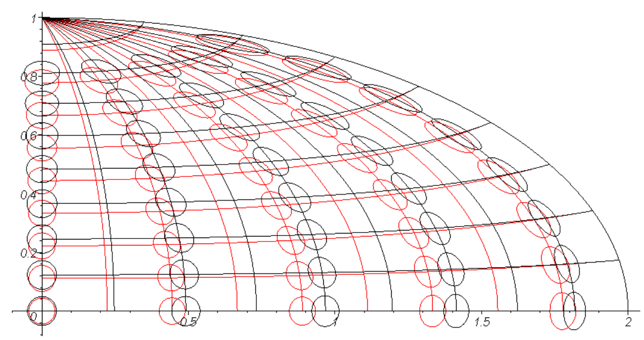

Figure 22 — Comparison of one quadrant of Modified Azimuthal projections: Aitoff (red), and Hammer Equal-Area (black).

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